Теор. Мат. Физика (2009), **161**(3), 346–366
Theor. Math. Phys. (2009), **161**(3), 1616–1633

A $\boldsymbol{\tau}$-FUNCTION SOLUTION TO THE SIXTH PAINLEVE TRANSCENDENT

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Abstract. We represent and analyze the general solution of the sixth Painlevé transcendent in the Picard–Hitchin–Okamoto class in the Painlevé form as the logarithmic derivative of the ratio of certain $\boldsymbol{\tau}$–functions. These functions are expressible explicitly in terms of the elliptic Legendre integrals and Jacobi θ-functions, for which we write the general differentiation rules. We also establish a relation between the $\mathcal{P}_6$ equation and the uniformization of algebraic curves and present examples.

Key words and phrases. Painlevé-6 equation, elliptic functions, theta functions, uniformization, automorphic functions.

Research supported by the Federal Targeted Program under contract 02.740.11.0238.

Contents

1. Introduction

The first appearance of the sixth Painlevé transcendent

$$\mathcal{P}_6: \quad y_{xx} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y_x + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} - \left(\delta - \tfrac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right) \tag{1}$$

dates back to the 1905 Fuchs' paper [9]. Contemporary applications of the equation are very well known [3, 25, 13] and the most nontrivial ones are in cosmology. Fuchs had already used the elliptic integral in his paper, and Painlevé a year later wrote this equation in the remarkable form

$$-\frac{\pi^2}{4}\frac{d^2z}{d\tau^2} = \alpha\wp'(z|\tau) + \beta\wp'(z-1|\tau) + \gamma\wp'(z-\tau|\tau) + \delta\wp'(z-1-\tau|\tau), \tag{2}$$

again using elliptic functions and performing the transcendental variable change $(y, x) \mapsto (z, \tau)$:

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \qquad y = \frac{1}{3} + \frac{1}{3}\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} - \frac{4}{\pi^2}\frac{\wp(z|\tau)}{\vartheta_3^4(\tau)} \tag{3}$$

(these formulae are equivalent to the original Painlevé ones; see relations (4) below). Here and hereafter, $\wp(z|\tau) := \wp(z|1,\tau)$ and other Weierstrass' functions $(\sigma, \zeta, \wp, \wp')$ [1, 8, 26]

are constructed with the half-periods $(\omega, \omega') = (1, \tau)$, and the Jacobi theta constants $\vartheta(\tau)$ depend on the modulus τ belonging to the upper half-plane $\mathbb{H}^+$, i.e. $\Im(\tau) > 0$ (see the appendix for definitions).

This Painlevé result, known as the $\wp$–form of the $\mathcal{P}_6$-equation was rediscovered and developed in the 1990s with other approaches, and the deep relation between (1) and second-order linear differential equations of the Fuchsian class is well known. The transition from these equations to Painlevé representation (2) was recently well described informally [2]. In the same place additional references can also be found.

The interest in general Painlevé equation (1) increased dramatically after Tod demonstrated in 1994 [25], that imposing the $\mathbb{SU}(2)$-invariance condition on the metric (the famous cosmological Bianchi-IX model) together with assuming the metric conformal invariance are consistent with the Einstein–Weyl self-duality equations $R_{ab} = \frac{1}{2}\Lambda g_{ab}$ and $W^{+}_{abcd} = 0$. These equations are partial differential equations but they can be reduced to an ordinary differential equation with respect to the conformal factor, which can be expressed in terms of a solution of the $\mathcal{P}_6$ equation for particular values of the parameters $(\alpha, \beta, \gamma, \delta)$ (sect. 2.1). Equation (1) with arbitrary parameters was later obtained as a self-similar reduction of the equations for the Ernst potential in general relativity [23]. This is not the only example of the appearance of equation (1) in applications: it suffices to recall the Yang–Mills equations (Mason–Woodhouse (1996)), two-dimensional topological field theories (Dubrovin), etc. In the general setting equation (1) was also derived in [17] as a self-similar reduction. See also references in these works.

1.1. **The Painlevé substitution (3).** Equation (1) and substitution (3) reflect the properties of not only the equation itself but also its solutions. If an equation is of the 2nd order, linear in y_{xx}, and rational in y_x and if its solutions have only fixed branching points (the Painlevé property), then the number of these points, as is known, is reducible to three. We can always choose them to be located at the points $x_j = \{0, 1, \infty\}$, and the equation then either takes Fuchs–Painlevé form (1) or represents some of its limit cases [20]. In turn, the plane of the variable x can be conformally mapped one-to-one to the fundamental quadrangle of the modular group $\mathbf{\Gamma}(2)$ [8] in the plane of the new variable τ using the modular function $x = k'^2(\tau)$, i.e. the first formula in (3). Function $k'^2(\tau)$ behaves exponentially with respect to the local parameter τ at the preimages of the points x_j. Because of this the branching $y = y(x)$ of an arbitrary power-law or logarithmic character in the vicinity of x_j transforms into a locally single-valued dependence on τ [4, §7]. This partially explains[1] the origin of substitution (3). Fuchs obtained this substitution [9] using the Legendre equation and Painlevé used the modular function [20].

Painlevé himself wrote the equation not in form (2) but in terms of the modular function $x = \varphi(\xi)$ with the three singular points $x_j = \{0, 1, \infty\}$, appearing when inverting the elliptic integral

[1] This is a partial explanation because a general rigorous statement on the branching of solutions of the $\mathcal{P}_6$ equation has apparently not yet been introduced in the literature. For example, it is unknown whether it is (at most) a local branching of the type of the rational function $x^\alpha \ln^n x$. See the important paper [12], which was devoted to precisely this question and where Fuchsian equation (5)—an equivalent to equation (2)—was the main object of study.

$$z=\frac{1}{\omega_1}\int_{\infty}^{y}\frac{dy}{\sqrt{y(y-1)(y-x)}},\qquad \frac{\omega_2}{\omega_1}=\xi,\qquad y=f(z,\xi). \tag{4}$$

In his original notation, the equation $\mathcal{P}_6$ in the $\wp$-form is as follows [20, p. 1117]:

$$y=f(z,\xi),\qquad z''_{\xi^2}=\frac{\varphi'(\xi)}{\varphi(\varphi-1)}\big[\alpha f'(\eta,\xi)-\beta f'(\eta+1,\xi)+\gamma f'(\eta+\xi,\xi)+(\tfrac{1}{2}-\delta)f'(\eta+1+\xi,\xi)\big].$$

where the misprint $\eta\mapsto z$ should be corrected. Equation (1), has the form of an 'inhomogeneous' Legendre differential equation [9]

$$\frac{d^2u}{dt^2}+\frac{2t-1}{t(t-1)}\frac{du}{dt}+\frac{u}{4t(t-1)}=\frac{\sqrt{\lambda(\lambda-1)(\lambda-t)}}{2t^2(t-1)^2}\left[k_\infty-k_0\frac{t}{\lambda^2}+k_1\frac{t-1}{(\lambda-1)^2}-k_t\frac{t(t-1)}{(\lambda-t)^2}\right], \tag{5}$$

where $u=\int_0^\lambda\frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}$ and $\lambda=\lambda(t)$ is a solution of the equation $\mathcal{P}_6$.

1.2. **The motivation for the work and the paper content.** Other singular points are movable but are poles, and the global behavior of solutions is therefore totally governed by the logarithmic derivatives of 'entire' functions [20], [7, pp. 77–180]. Painlevé stressed that construction of such transcendental functions completes the procedure of integrating an equation (the concept of the 'intégration parfaite' in the Painlevé terminology). The pole character is known for all Painlevé equations [11, 7] and all their solutions except those for the equation $\mathcal{P}_1$ have the structure [20, p. 123]

$$y\sim\frac{d}{dx}\mathrm{Ln}\frac{\boldsymbol{\tau}_2}{\boldsymbol{\tau}_1} \tag{6}$$

(in the generic case) with the functions $\boldsymbol{\tau}_{1,2}(x)$ having no movable singularities. See the works [18, p. 371] and [7, p. 165] for the complete list of such formulae.

If we disregard the known automorphisms of solutions of the equation $\mathcal{P}_6$ in the space of the parameters $(\alpha,\beta,\gamma,\delta)$ (see, e.g., the Gromak paper in [7], [15], [11, §42], [14] and the references therein), only two cases are known where the general integral for equation (1) can be written. These are the solutions of Picard [22, pp. 298–300] and Hitchin (sect. 2.1), but Painlevé form (6) is known for neither of these solutions.

On the other hand, representing solutions in terms of the corresponding analogues of $\boldsymbol{\tau}$-functions occurs in almost all investigations of nonlinear equations somehow related to the integrability property: in the Hirota method, in the theory of isomonodromic deformations [15], in soliton theory, and in its Θ-functional generalizations. In this respect, it is interesting to show that the Painlevé equations also admit a nontrivial situation in which the functions $\boldsymbol{\tau}_{1,2}(x)$ can be written explicitly when we have the general integral of motion. Here, we fill this gap (sects. 2.2, 4) and also describe the distribution of solution singularities (sect. 3). Technically, we need special differential properties of Jacobi's theta-functions and their theta-constants. These properties are also new, and we present them in the appendix. As follows from the solution (this was properly noted in [15]), the character of the Painlevé $\boldsymbol{\tau}$-functions differs drastically from that of the Θ-functional solutions of the solitonic equations. Nevertheless, a variant of the straight-line section of a manifold is

present (sect. 4). In section 5 we demonstrate the connection of the equation $\mathcal{P}_6$ with the uniformization of nontrivial algebraic curves. To what was said in sect 1.1, we add that Painlevé himself explicitly wrote [7, p. 81] about representing solutions in terms of single-valued functions in the context of the Painlevé property. Picard mentioned the same thing many times in [21, pp. 93, 188], [22]. The relation to uniformization mentioned below and the presentation of numerous examples (sect. 5.2) is therefore not accidental.

2. The Painlevé form

2.1. The Hitchin solution. The Hitchin solution corresponds to the parameter values

$$\alpha = \beta = \gamma = \delta = \frac{1}{8}, \tag{7}$$

which were found in [13] under the condition that the Einstein equation metric is conformal with $\mathbb{SU}(2)$-invariant anti-self-dual Weyl tensor [25]. Equation (2) in this case can be easily reduced to the equivalent form written in theta functions

$$\frac{d^2 z}{d\tau^2} = 4\pi \boldsymbol{\eta}^9(\tau)\,\frac{\theta_1(2z|\tau)}{\theta_1^4(z|\tau)}\,, \tag{8}$$

where $\boldsymbol{\eta}$ is the Dedekind function

$$\boldsymbol{\eta}(\tau) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}\tau}\prod_{k=1}^{\infty}\bigl(1 - \mathrm{e}^{2k\pi\mathrm{i}\tau}\bigr).$$

The general solution to equation (8) in parametric form (3) is [13]

$$\wp(z|\tau) = \wp(A\tau + B|\tau) + \frac{1}{2}\,\frac{\wp'(A\tau + B|\tau)}{\zeta(A\tau + B|\tau) - A\eta'(\tau) - B\eta(\tau)}\,, \tag{9}$$

where A, B are the integration constants. This solution in a different implicit form was mentioned by Okamoto [19, p. 366].

Solution (9) is not only the most nontrivial of all those currently known but is also rather instructive for $\mathcal{P}$-equations in general because it provides the general integral. It has been considered in detail in the literature and is certainly more than 'just a solution'. The Picard solution is degenerate in this respect because it is a perfect square (see sect. 4.1). This explains why we choose the Hitchin solution for further analysis. Other particular cases are known that contain variations of the Painlevé ∂_xLn-form but do not contain $\boldsymbol{\tau}$-functions. Allowing more freedom in choosing the parameters $(\alpha, \beta, \gamma, \delta)$, these solutions nevertheless contain only one integration constant and are related to the linear hypergeometric ${}_2F_1$-equation [7, p. 733], [11, §44], [19, p. 374 §5.4], [14, p. 145].

2.2. The Painlevé form. One of the theta-function versions of solution (9) was proposed in [13] and another form in [15]. But the complexity of these solution forms was mentioned in [13, p. 75] itself, in [15, p. 901], and also in work [3], consisting entirely of calculations. In those papers, solutions were also presented in the parametric forms and contained the set of functions $\theta, \theta', \theta'', \theta''', \vartheta, \vartheta', \vartheta'''$. However one can obtain the solution in a compact form if we transform Weierstrass functions in (9) into the theta functions (cf. [13, p. 74]).

Proposition 1. *The general solution to equation (1) with parameters (7) is*

$$y = \frac{\sqrt{x}}{\theta_1^2}\left\{\frac{\pi\vartheta_2^2{\cdot}\theta_2\theta_3\theta_4}{\theta_1' + \pi\mathrm{i}A\theta_1} - \theta_2^2\right\}, \tag{10}$$

where $\theta = \theta\big(\tfrac12 A\tau + \tfrac12 B\big|\tau\big)$ *and* $\vartheta = \vartheta(\tau)$.

The possibility of representing solution (10) in explicit form (6) is related to nontrivial properties of the theta functions, for instance, to the identity that follows from Lemma 3 in the appendix

$$\frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\big\{\zeta(z|\tau) - z\eta(\tau)\big\} = \wp(z|\tau) + \frac12\frac{\wp'(z|\tau)}{\zeta(z|\tau) - z\eta(\tau)} + \eta(\tau),$$

Comparing this property with (9), we obtain the total logarithmic derivative,

$$\wp(z|\tau) = \frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\frac{\zeta(A\tau + B|\tau) - A\eta'(\tau) - B\eta(\tau)}{\boldsymbol{\eta}^2(\tau)}. \tag{11}$$

Replacing $(A, B) \mapsto 2(A, B)$ and transforming the right-hand side of (11) into θ-functions (lemma 1 sect. 7) we can write the solution in the form

$$y = \frac{2\mathrm{i}}{\pi}\frac{1}{\vartheta_3^4(\tau)}\frac{d}{d\tau}\mathrm{Ln}\frac{\theta_1'(A\tau + B|\tau) + 2\pi\mathrm{i}A\theta_1(A\tau + B|\tau)}{\vartheta_2^2(\tau)\theta_1(A\tau + B|\tau)}, \qquad x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}. \tag{12}$$

To rewrite the solution in the initial variables (x, y), we must use the formulae inverting substitution (3) and its differential. They can be written in terms of the complete elliptic integrals K, K' [1, 27]. It is easy to see that this transition together with the differentiation is determined by the formulae

$$\frac{d}{d\tau} = \pi\mathrm{i}x(x-1)\vartheta_3^4(\tau)\frac{d}{dx}, \qquad \vartheta_2^2(\tau) = \frac{2}{\pi}\sqrt{1-x}\,K'(\sqrt{x}), \tag{13}$$

and inversion of the first formula in (3) is given by the expression (series)

$$\mathrm{e}^{\pi\mathrm{i}\tau} = \exp\left\{-\pi\frac{K(\sqrt{x})}{K'(\sqrt{x})}\right\} = \tag{14}$$

$$= \Big(\frac{1-x}{16}\Big) + 8\Big(\frac{1-x}{16}\Big)^2 + 84\Big(\frac{1-x}{16}\Big)^3 + 992\Big(\frac{1-x}{16}\Big)^4 + 12514\Big(\frac{1-x}{16}\Big)^5 + \cdots.$$

Simplifying the obtained expressions and replacing $\mathrm{i}A \mapsto A$, we obtain the sought solution form.

Theorem. *The general integral of equation (1) with parameters (7) has the form*

$$y = 2x(1-x)\frac{d}{dx}\mathrm{Ln}\frac{\theta_1'\big(A\frac{K}{K'} + B\big|\frac{\mathrm{i}K}{K'}\big) + 2\pi A\cdot\theta_1\big(A\frac{K}{K'} + B\big|\frac{\mathrm{i}K}{K'}\big)}{\sqrt{1-x}\,K'\cdot\theta_1\big(A\frac{K}{K'} + B\big|\frac{\mathrm{i}K}{K'}\big)}. \tag{15}$$

Here and occasionally hereafter, we use the shorthand notation $K := K(\sqrt{x})$ *and* $K' := K'(\sqrt{x})$.

The quantities K, K' can be written in terms of the hypergeometric functions [1, 8]

$$K(\sqrt{x}) = \frac{\pi}{2}\cdot{}_2F_1\Big(\frac12, \frac12; 1\Big|x\Big), \qquad K'(\sqrt{x}) = \frac{\pi}{2}\cdot{}_2F_1\Big(\frac12, \frac12; 1\Big|1-x\Big),$$

but we can use these series simultaneously only in the intersection of the convergence domains. In applied problems, we therefore avoid using numerous transformations for

analytic continuations of the ${}_2F_1$-series by regarding the functions K, K' as the complete elliptic integrals

$$K(\sqrt{x}) = \int\limits_0^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-x\lambda^2)}}, \qquad K'(\sqrt{x}) = \int\limits_x^1 \frac{d\lambda}{\sqrt{(\lambda^2-x)(1-\lambda^2)}}$$

(which are everywhere finite functions) or as the Legendre functions $P_{-\frac{1}{2}}$, $Q_{-\frac{1}{2}}$[27].

Verifying this by substituting (15) directly in (1) is a rather nontrivial exercise in which we must use Lemma 1 and its simple corollaries. We have to use also the known differentiation rules for complete elliptic integrals [1]. Analogously setting $E := E(\sqrt{x})$ and $E' := E'(\sqrt{x})$, we can write these rules as

$$\begin{aligned} 2\frac{d}{dx}K &= \frac{E}{x(1-x)} - \frac{K}{x}, & 2\frac{d}{dx}K' &= \frac{E'}{x(x-1)} - \frac{K'}{x-1}, \\ 2\frac{d}{dx}E &= \frac{E}{x} - \frac{K}{x}, & 2\frac{d}{dx}E' &= \frac{E'}{x-1} - \frac{K'}{x-1} \end{aligned} \tag{16}$$

and they imply the Legendre relation

$$\eta' = \tau\eta - \frac{\pi}{2}\mathrm{i} \quad \Leftrightarrow \quad EK' + E'K - KK' = \frac{\pi}{2}.$$

Speaking more rigorously, in (15), we must cancel the common 'non-entire' factor $\exp\left(\frac{1}{4}\pi\mathrm{i}\tau\right)$ in the series for θ_1 and θ_1'. But this factor and also the nonmeromorphic singularity of the solution determined by the factor $K'(\sqrt{x})$ are 'fixed'. Relations (15) and (16) demonstrate that the solution contains an additive term, which has a fixed critical singularity of the logarithmic type.

Corollary 1. *The function $y(x)$ and the solution $z(\tau)$ of equation (8) have the forms*

$$y = \frac{E'}{K'} + 2x(1-x)\frac{d}{dx}\mathrm{Ln}\left\{\frac{\theta_1'}{\theta_1}\Big(A\tfrac{K}{K'} + B\Big|\tfrac{\mathrm{i}K}{K'}\Big) + 2\pi A\right\}, \tag{17}$$

$$z = \wp^{-1}\!\left(\frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\left\{\frac{\theta_1'(A\tau+B|\tau)}{\theta_1(A\tau+B|\tau)} + 2\pi\mathrm{i}A\right\} - \eta(\tau)\Big|\tau\right), \tag{18}$$

where the elliptic integral $\wp^{-1}$ here and hereafter is defined as

$$\wp^{-1}(u|\tau) := \int\limits_\infty^u \frac{d\lambda}{\sqrt{4\lambda^3 - g_2(\tau)\lambda - g_3(\tau)}}.$$

To follow the meromorphic x-component (the movable poles) more closely, we substitute expansions of type (14) in formulae (15), (17) and in the theta-function series in the appendix. Weierstrass derived analogous series in relation to the solution of the modular inversion problem for the function $k^2(\tau)$ [26, pp. 53–4, 56, 58], [27, p. 367]. Explicit expansions in the vicinities of all other points can be written using the formulae in the appendix. Formulae (12), (15), (17) themselves provide an analytic answer for the power-law expansions in [13] on pages 89–92 and 108–9.

3. The pole distribution

One of the two series of poles x_{mn} of solution (15) admits an explicit parameterization,

$$\theta_1\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})}+B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)=0 \quad\Rightarrow\quad x_{mn}=\frac{\vartheta_4^4}{\vartheta_3^4}\Big(\frac{m-B}{n+A}\Big), \quad n,m\in\mathbb{Z}. \tag{19}$$

We must supplement it with the natural restriction on (n,m) of the form $\Im\big(\frac{m-B}{n+A}\big)>0$, which imposes no restrictions on A and B other than these numbers cannot be real simultaneously. Although this pole distribution is described by the simple formula (19), its actual form is rather involved (see Fig. 1).

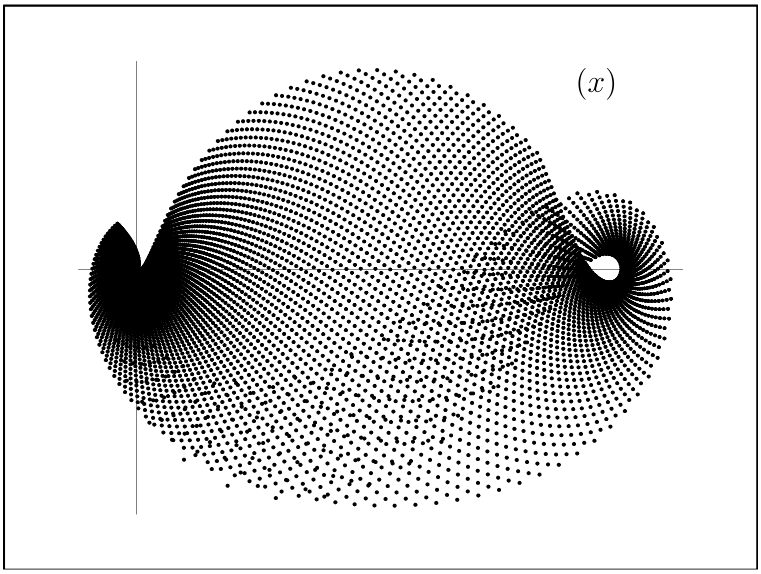

Figure 1. The poles x_{mn} of form (19) for solutions (15), (22) under $A=125.45-103.29\mathrm{i}$, $B=36.710-69.980\mathrm{i}$ and $(n,m)=-30\ldots70$.

The general and characteristic behavior of the 'orthogonal' directrices of the parameterizations $n=\text{const}$ and $m=\text{const}$, i.e. the pattern in Fig. 1 as a whole is shown in Fig. 2. The pattern in this figure can be interpreted as an analogue, although distant, of the two straight-line orbits of poles of any double periodic function. The pole distributions are deformed when the initial data (A,B) are changed, which generates various patterns, but we observe that all the poles concentrate near the fixed singularities $x_j=\{0,1,\infty\}$, as should be the case. We thus obtain the character of this deformation. If we take one pole $p=x_{mn}$ under fixed (m,n), then we find that it moves without changes along the straight-line characteristic $(n+A)t-(m-B)=0$ in the space of the initial data (A,B). The dynamics $p=p(t)$ for any pole of series (19) is given by the same function $p(t)=\frac{\vartheta_4^4}{\vartheta_3^4}(t)$ but with times t_k that differ by a fractional linear transformation for each pole $p_k=p(t_k)$. The function $p(t)$ and the structure of the fundamental domain of its automorphism group are well studied. Real or purely imaginary poles can also be easily described.

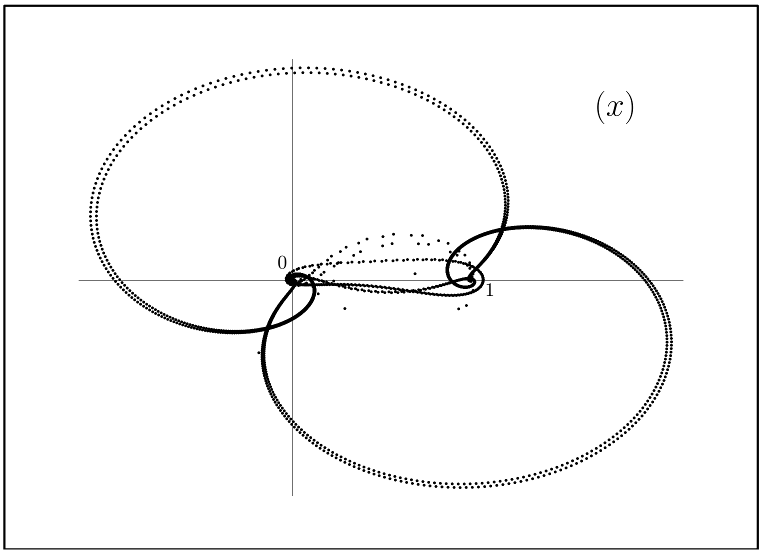

FIGURE 2. Two pairs of directrices $n = (12, 13)$ and $m = (129, 130)$ of pole distributions (19) for $A = -195.45 + 103.29\mathrm{i}$, $B = -6.710 + 79.98\mathrm{i}$. All the seemingly 'casual' points are also 'good' poles belonging to these directries.

We further note that the Okamoto transformation [11, §47], [15]

$$
\begin{aligned}
\text{PICARD}_y \mapsto \text{HITCHIN}_{\tilde{y}}: \qquad & \tilde{y} = y + \frac{y(y-1)(y-x)}{x(x-1)y_x - y^2 + y}, \\
\text{HITCHIN}_y \mapsto \text{PICARD}_{\tilde{y}}: \qquad & \tilde{y} = y - \frac{y(y-1)(y-x)}{x(x-1)y_x + \frac{1}{2}y^2 - xy + \frac{1}{2}x},
\end{aligned}
\tag{20}
$$

which is known to transform the Picard and Hitchin solutions one into the other, leaves poles (19) invariant but creates or, under the inverse transformation, annihilates the second series of poles determined by roots of the equation

$$\zeta(A\tau + B|\tau) = A\zeta(\tau|\tau) + B\zeta(1|\tau).$$

Solving this equation is equivalent to finding the A-points of the transcendental function

$$f(x; A, B) = \frac{1}{2\pi} \frac{\theta_1'\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} - B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)}{\theta_1\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})} - B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)}, \tag{21}$$

i.e. to solving the equation $f(x; A, B) = A$. The pole series determined by relation (21) has a much more involved description and evolution than for pole series (19).

4. THE $\boldsymbol{\tau}$-FUNCTIONS

4.1. The Picard solution. Pole series (19) also exhausts all the poles of solutions to equation (1) in the Picard case $\alpha = \beta = \gamma = \delta = 0$. It then follows from (2) that

$\ddot{z}=0 \;\Rightarrow\; z=a\tau+b$. Writing relations (3) in terms of the theta function, we obtain another form of substitution (3) and the equivalent form of the Picard solution in [22]:

$$y=-\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}\frac{\theta_2^2\big(\frac{z}{2}|\tau\big)}{\theta_1^2\big(\frac{z}{2}|\tau\big)} \qquad\Rightarrow\qquad y_{\text{PIC}}=-\sqrt{x}\,\frac{\theta_2^2\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})}+B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)}{\theta_1^2\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})}+B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)}. \tag{22}$$

A comment is in order. Strictly speaking, the original Picard solution u_{PIC} [22, p. 299] and the Fuchs–Painlevé representation (1) are not identical but are related by an inverse transformation. Picard constructed an example of the equation using the Legendre form $\Delta x=\sqrt{(1-x^2)(1-k^2x^2)}$, while Painlevé used Riemann form (4). The transition between solutions of these equations is therefore given by the formula

$$u_{\text{PIC}}\big(\tfrac{1}{x}\big)=2\,\frac{\sqrt{e-\wp}}{\pi\vartheta_3^2}=\sqrt{y_{\text{PIC}}(x)}. \tag{23}$$

In other words, we have the property that the case of parameters $\alpha=\beta=\gamma=\delta=0$ is a case (not unique) of Painlevé equation (1) such that the square root of any of its solution satisfies, in turn, an equation that also has the Painlevé property, namely, the Picard equation [22, p. 299]

$$\frac{d^2u}{dx^2}-\Big(\frac{du}{dx}\Big)^2\frac{u(2xu^2-1-x)}{(1-u^2)(1-xu^2)}+\frac{du}{dx}\bigg[\frac{u^2-1}{(1-x)(1-xu^2)}+\frac{1}{x}\bigg]-\frac{u(1-u^2)}{x(1-x)(1-xu^2)}=0, \tag{24}$$

where we must amend the last term with the omitted factor $\frac{1}{4}$. This equation appears at the end of the almost two-hundred-page-long Picard treatise [22, p. 298] as équation différentielle curieuse satisfied by the function $u=\operatorname{sn}(a\omega+b\omega')$, regarded as a function of the elliptic modulus k^2 $(=x)$. Therefore, the ansatz for studying the branching of solutions to (1) assumed in [12]

$$y(x)=\wp\big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\omega_1(x),\omega_2(x)\big)+\frac{1+x}{3},$$

contains a somewhat 'redundant' perfect square and can be conveniently replaced with a structure of Hitchin's type (15), (17), which simplifies the analysis.

Remark 1. In addition to the Picard solution, all its transformations obtained by actions of discrete symmetries of equation (1) [11, §42], for instance, by the variable change

$$y\mapsto\widetilde{y}=\frac{y-x}{y-1}=\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}\frac{\theta_3^2\big(A\tau+B|\tau\big)}{\theta_4^2(A\tau+B|\tau)},$$

are perfect squares. Adding the modular substitutions from the group $\mathbf{\Gamma}(1)$ and the anharmonic group S_3, transforming the variable x [1, §23], we obtain the complete set of transformations in [11, pp. 214–5] and changes of the parameters $(\alpha,\beta,\gamma,\delta)$. Together with the Okamoto transformations, all the transformations obtained above and below expressed in terms of the theta functions and $\theta,\boldsymbol{\tau}$-functions reduce to permutations of indices of the functions ϑ,θ,θ' in formulae (10), (15), (17), and (18). By the Picard–Hitchin solution, we therefore understand the whole class of these solutions, without especially mentioning this. The Picard–Hitchin solutions are functions of $\sqrt{x}$. On the other hand, the quantity $\sqrt{x}=s$ itself is related to the second-order linear differential Fuchs equation via the integrals $K,K'(\sqrt{x})$ and substitution (3). This equation has four singularities located at the points $s=\{0,1,-1,\infty\}$ because the integrals K and $K'(x)$ satisfy the Legendre equation with the three singularities $x_j=\{0,1,\infty\}$. We thus obtain the particular case of the Heun equation

$$Y_{ss}=-\frac{1}{2}\frac{(s^2+1)^2}{s^2(s^2-1)^2}Y \qquad\Rightarrow\qquad \frac{Y_2}{Y_1}=\mathrm{i}\,\frac{K(s)}{K'(s)}=\tau. \tag{25}$$

The character of quadrature (15) is therefore described by the following statement.

Corollary 2. *Painlevé equation* (1) *in the Picard and Hitchin cases is integrable in* (θ', θ)*-function quadratures over the differential field determined by the Heun functions of form* (25) *or, equivalently, by the hypergeometric Legendre functions.*

4.2. **One-parameter solutions.** We note that neither formula (9) nor the first formula in (22) imply in any obvious way that the second-order poles from the Picard series become the first-order poles for (17). Analogously comparing formulae (3), (9), and (22), we find that the first formula in (22) does not provide the solution behavior in the limit cases for the parameters (A, B), while this behavior follows automatically from formula (17). For instance, we set $A\tau + B \to 0$ with $B/A = \alpha$ in (17) and (18). Taking (14) and (16) into account, we then immediately obtain the α-parameter family of solutions

$$
\begin{aligned}
y = \frac{\alpha E' - E + K}{\alpha K' + K} \Leftrightarrow z &= \wp^{-1}\!\left(\frac{\pi \mathrm{i}}{2}\frac{d}{d\tau}\mathrm{Ln}\big\{(\tau+\alpha)\boldsymbol{\eta}^2(\tau)\big\}\Big|\tau\right)\\
&= \wp^{-1}\!\left(\frac{1}{2}\frac{\pi \mathrm{i}}{\tau+\alpha} - \eta(\tau)\Big|\tau\right),
\end{aligned}
$$

which can be usefully compared with the corresponding Hitchin formula in [13, p. 78].

Another case looks more complicated, and Hitchin considered it only in the expansion form because it results in a singular conformal structure. This case is more conveniently analyzed in form (11), where it corresponds to the limits $A\tau + B \to \big\{1, \tau, \tau+1\big\}$. Setting $A\tau + B = \varepsilon\tau + (1 + \varepsilon\alpha)$ and letting ε tend to zero, we obtain the solution for which we preserve the $\boldsymbol{\tau}$-form (6):

$$
\begin{aligned}
\wp(z|\tau) &= \frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\frac{2(\tau+\alpha)\dot{\vartheta}_2 + \vartheta_2}{\boldsymbol{\eta}^2\vartheta_2}\\
&= \frac{\pi^2(\vartheta_3^4+\vartheta_4^4)(\eta'+\alpha\eta) + (\tau+\alpha)\big(g_2 - \frac{1}{2}\pi^4\vartheta_3^4\vartheta_4^4\big)}{\pi^2(\vartheta_3^4+\vartheta_4^4)(\tau+\alpha) + 12(\eta'+\alpha\eta)}\,.
\end{aligned}
\tag{26}
$$

Here and hereafter, we let the dot above a symbol denote the derivative with respect to τ and omit the argument τ of the functions $\boldsymbol{\eta}, \eta, \eta', g_2, \vartheta$ for brevity. Moreover, we can replace g_2 with its ϑ equivalent (see appendix). We can calculate this solution in the variables (x, y) if we supplement relations (13), (14), (16) with additional relations to obtain the complete basis of the transition between the 'τ-' and 'x-representations'. We can do this using the expressions for the quantities ϑ, $\dot{\vartheta}_2$ and η:

$$
\vartheta_3^2 = \frac{2}{\pi}K', \qquad \dot{\vartheta}_2 = \frac{\mathrm{i}}{\pi}\vartheta_2 K'E', \qquad \eta = K'E' - \frac{x+1}{3}K'^2.
$$

Simplifying and replacing $\alpha \mapsto \mathrm{i}\alpha$, we obtain the compact solution

$$
y = \frac{x(\alpha K' + K)}{\alpha E' - E + K}.
$$

We can analogously simplify the two other limit cases:

$$
y = \frac{2(\alpha K' + K)(E' - K') - \pi}{2(\alpha K' + K)(E' - xK') - \pi}\,x \quad \Leftrightarrow \quad \wp(z|\tau) = \frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\frac{2(\tau + \mathrm{i}\alpha)\dot{\vartheta}_3 + \vartheta_3}{\boldsymbol{\eta}^2\vartheta_3},
$$

$$y = \frac{2(\alpha K' + K)(E' - xK') - \pi}{2(\alpha K' + K)(E' - K') - \pi} \qquad \Leftrightarrow \qquad \wp(z|\tau) = \frac{\pi}{2\mathrm{i}}\frac{d}{d\tau}\mathrm{Ln}\frac{2(\tau + \mathrm{i}\alpha)\dot{\vartheta}_4 + \vartheta_4}{\boldsymbol{\eta}^2\vartheta_4}.$$

We see that these solutions are mutually related by transformations of the form $y \mapsto \frac{x}{y}$; they therefore cannot originate from the modular transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -\frac{1}{\tau}$. But we can easily see that they reflect discrete symmetries of equation (8) of the form $z \mapsto \{z+1, z+\tau\}$, and this entails the above transformation for y, which is easily recognizable when written in the standard $\wp(z|\omega, \omega')$-representation:

$$\wp(z+\omega) - e = \frac{(e-e')(e-e'')}{\wp(z) - e} \qquad \Leftrightarrow \qquad \widetilde{y} = \frac{x}{y}.$$

Of course, we can apply this symmetry to any Picard–Hitchin solution. We also note that we need the pair of functions (E, E') not only as a formal completion of the set of functions (K, K'): both the Hitchin and the Chazy solutions obtained in [16] and also any η, ϑ, θ-function solutions to equation (1) are expressed in terms of (E, E').

4.3. **The $\boldsymbol{\tau}$-functions.** We also presented examples of the Painlevé $\partial_x\mathrm{Ln}$-forms for the degenerate Hitchin solutions because the $\boldsymbol{\tau}$-functions are important for the theory of Painlevé equations, being the generators of their Hamiltonians and other objects [14, 7, 19, 18]. But the $\boldsymbol{\tau}$-functions are primarily important not for constructing Hamiltonians, which are known for all Painlevé equations, but for representing solutions in form (6) see [18] and the explanations in [7, pp. 740–1] in regard to constructing $\boldsymbol{\tau}$-functions corresponding to the positive and negative residue series for solutions of the equation $\mathcal{P}_2$.

We note that other known examples (see the citations at the end of sect. 2.1) require the existence of the explicit transition ${}_2F_1 \mapsto \tau$ because the ${}_2F_1$-series are essentially nonholomorphic functions with branching. Reducing them to the 'genuine' (τ)-form is a very complicated problem, which is equivalent to inverting the integral ratios for equations of the Fuchsian type. In this respect, formulae are always 'final' in the uniformizing τ-representation (sect. 5), while the transition ${}_2F_1 \mapsto \tau$ is nontrivial and may contain several hypergeometric series. For instance, in the above examples, besides the integrals K, K', E, E', i.e. functions of the type ${}_2F_1\big(\pm\frac{1}{2}, \frac{1}{2}; 1\big|z\big)$, we encounter the function $\boldsymbol{\eta}^2(\tau)$. It can be shown [4, p. 263] that the hypergeometric equation related to this function is just the equation for the function ${}_2F_1\big(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}\big|J\big)$, i.e. the equations that determine the classic Klein modular invariant $J = (g_2^3/\Delta)(\tau)$ [1, §10], [8, §14.6.2]:

$$J(J-1)\frac{d^2}{dJ^2}\boldsymbol{\eta}^2 + \frac{1}{6}(7J-4)\frac{d}{dJ}\boldsymbol{\eta}^2 + \frac{1}{144}\boldsymbol{\eta}^2 = 0.$$

By this means we currently have only the Hitchin class in which all the results including degenerations can be written explicitly. It remains to note that the found Painlevé form admits a natural interpretation, analogous to formulae of the straight-line sections of the two-dimensional Jacobians in soliton equation theory. Function (21) is the canonically normalized meromorphic elliptic integral $I(z - B|\tau)$, which has only the first-order pole at the point $z = B$ and is considered on the straight-line section $z - A\tau + B = 0$ of the space $\{\mathbb{C} \times \mathbb{H}^+\}$, obtained as the direct product of the elliptic curve (the parameter z) and the one-dimensional moduli space of elliptic curves (the parameter τ).

The $\boldsymbol{\mathcal{T}}$-function constructed in [15], corresponds only to Picard poles (19) and coincides with the function

$$\boldsymbol{\mathcal{T}}_1(x;A,B)=\theta_1\Big(A\frac{K(\sqrt{x})}{K'(\sqrt{x})}+B\Big|\frac{\mathrm{i}K(\sqrt{x})}{K'(\sqrt{x})}\Big)\,.$$

up to a fixed nonmeromorphic factor. Hence, there exists a second Hamiltonian corresponding to the second $\boldsymbol{\mathcal{T}}$-function and poles with the opposite signs of residues. This function is given by the formula

$$\boldsymbol{\mathcal{T}}_2(x;A,B)=\boldsymbol{\mathcal{T}}_1(x;A,B)\frac{d}{dB}\mathrm{Ln}\big\{\boldsymbol{\mathcal{T}}_1(x;A,B)\mathrm{e}^{2\pi AB}\big\}.$$

We choose such a representation because meromorphic Abelian integrals (integrals of the second kind) of the type of functions (21) can be represented as derivatives of logarithmic integrals (integrals of the third kind) with respect to the parameter determining the location of one of the two logarithmic poles (with respect to the constant B). This may entail further generalizations, but we do not discuss this here.

5. The equation $\mathcal{P}_6$ and uniformization of curves

5.1. **Algebraic solutions.** Having the general integral leads to several corollaries, for instance, to the limit cases with respect to the parameters (A,B), for example,

$$A\tau+B=\frac{n}{N}\tau+\frac{m}{N} \tag{27}$$

We thus obtain an infinite series of algebraic solutions of the form $P(x,y)=0$, where P is a polynomial in its arguments. Hitchin also presented particular solutions of the algebraic type, and the origin of the whole algebraic Picard series was revealed in [16] where asymptotic regimes and monodromies related to this and other solutions and the character of their branching were also considered.

It is less known that some of these results and also the very scheme for constructing such solutions were already contained in Fuchs' paper [24] with the same title as his well-known paper[2] of 1907. The procedure for constructing solutions was missed in [10] but the method for obtaining these solutions is obviously the same: the theorems of multiplication and addition for elliptic functions can be used. Moreover, Fuchs pointed out [10] how to use the known Kiepert determinant multiplication formula for $\wp(Nz)$ as a rational function of $\wp(z)$ [26, p. 19], [27, p. 332]; he also wrote the Puiseux series for these solutions [10, §IV] and considered the monodromy of the Fuchs equations in great detail. Such determinant schemes can be interesting in themselves because the list of algebraic solutions of the Painlevé equations steadily increases. We restrict ourself to the statement (without proof) that we can obtain effective recursion schemes for constructing such solutions. We do not present examples of these recursions because another immediate corollary from what was said above seems more interesting to us.

From the explanations in sect. 1.1 and from explicit form (18) of the $z(\tau)$-dependence for the solution to equation (2), we obtain infinite series of algebraic curves $P(u,v)=$

[2] In a one-line footnote in [22, p. 300] Picard also claimed that the infinite series of algebraic solutions appears in the case where the numbers (A,B) are commensurable.

0 of nontrivial genera $g > 1$ with the explicit parameterizations by the single-valued analytic functions $u(\tau), v(\tau)$. Because the Kiepert formula above contains a redundant set of functions $(\wp, \wp', \wp'', \ldots)$, it is more convenient to pass to the language of theta functions, which also provides a more compact answers. In this case, it is desirable to have multiplication theorems directly for the functions θ_k, θ_1'. To the best of our knowledge, such formulae are absent in the literature, and we therefore present them in the appendix (Lemma 2). For the Hitchin solutions, we can use transformation (20), but we lack the efficiency when acting with them on algebraic functions whose complexity rapidly increases as N increases. It is therefore more convenient to use the abovementioned theorems for the θ, θ'-functions directly in formulae (12), (18) (also see Proposition 2 below). It is obvious how we can apply these theorems to the solutions under investigation, and we therefore pass to demonstrating specific examples and their corollaries.

5.2. **Examples.** Setting $A\tau + B = \frac{1}{3}$, we reconstruct the Dubrovin solution [16, (4.5)]

$$y^4 - (6y-4)yx + (4y-3)x^2 = 0. \tag{28}$$

Formula (3) then results in the curve of genus $g = 3$ of the form

$$4(u^2 + u^{-2})v = v^4 - 6v^2 - 3, \qquad \left\{ u = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}, \quad v = \frac{\theta_2^2\left(\frac{1}{6}\middle|\tau\right)}{\theta_1^2\left(\frac{1}{6}\middle|\tau\right)} \right\} \tag{29}$$

if we use the substitution $(x = u^4,\ y = -u^2 v)$. This curve does not belong to the series of classic modular equations [8, §14.6.4] nor to their closest variations [4, §7]. In turn, it is remarkable that because of the birational transformation

$$\left\{ \begin{aligned} u &= \frac{\boldsymbol{x}^4 - \boldsymbol{y} - 1}{4\boldsymbol{x}} \\ v &= \frac{\boldsymbol{x}^4 - \boldsymbol{y} + 1}{2\boldsymbol{x}^2} \end{aligned} \right. , \qquad \left\{ \begin{aligned} \boldsymbol{x} &= \frac{u(v^2-1)}{2(u^2+v)} \\ \boldsymbol{y} &= \frac{(v^2+3)^2}{v^2(v^2-1)} \left(\frac{4u^2 v}{v^2+3} + 1 \right) \end{aligned} \right. ,$$

equation (29) is isomorphic to the classic hyperelliptic Schwarz curve

$$\boldsymbol{y}^2 = \boldsymbol{x}^8 + 14\boldsymbol{x}^4 + 1. \tag{30}$$

In Schwarz's Gesammelte Abhandlungen [24] this curve appears in many contexts in both volumes[3], but no variants of its parametrization were known. We present the formula for $\boldsymbol{x}(\tau)$, because it admits the compact simplification

$$\boldsymbol{x} = \frac{\boldsymbol{\eta}^3(\tau)\,\theta_2\left(\frac{1}{3}\middle|\tau\right)}{\theta_1^2\left(\frac{1}{6}\middle|\tau\right)\theta_3^2\left(\frac{1}{6}\middle|\tau\right)}. \tag{31}$$

For this curve and other curves and functions introduced by this method, we can write the remaining uniformization attributes (which is a purely technical problem): differential equations for function $y(\tau)$, the ${}_2F_1$-solutions of the corresponding Fuchsian equations, the representations of the monodromy group of these equations, the Poincaré polygons

[3] For instance, it appears in the 1867 paper [24, **I**: p. 13], where the famous Schwarz derivative $\boldsymbol{\Psi}(s, u)$ was introduced.

[8], the Abelian integrals as functions of τ, their reductions, if any, to elliptic integrals, etc.

If we consider analogous solution (9), with the same parameters $A\tau + B = \frac{1}{3}$, then we obtain the curve that is the Hitchin solution

$$y^4 - 2(x+1)(y^2+x)y + 6xy^2 + x^3 - x^2 + x = 0. \tag{32}$$

But this curve with the substitution $x = u^4$ must become isomorphic to the previously described curves because the initial Picard (28) and Hitchin (32) solutions have the genus $g = 0$. In turn, formulae (3) and (9) provide an equivalent but not obvious parameterization of (32)

$$y = \frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)} \left(\pi \vartheta_2^2(\tau) \frac{\theta_3\theta_4}{\theta_2\theta_1'} \big(\tfrac{1}{6}\big|\tau\big) - 1 \right) \frac{\theta_2^2}{\theta_1^2} \big(\tfrac{1}{6}\big|\tau\big),$$

which follows from the nontrivial ϑ, θ, θ'-function relations[4]. The general statement about the algebraic Hitchin solutions follows from formulae (3) and (10).

Proposition 2. *The algebraic Hitchin solutions admit the parameterization*

$$x = \frac{\vartheta_4^4}{\vartheta_3^4}, \qquad y = \frac{\vartheta_4^2}{\vartheta_3^2}\frac{\theta_2}{\theta_1^2}\left\{ \frac{\pi\vartheta_2^2\theta_3\theta_4}{\theta_1' + 2\pi\mathrm{i}\frac{n}{N}\theta_1} - \theta_2 \right\}, \tag{33}$$

where the θ, θ'-functions are taken with the arguments in (27). Functions (33) satisfy autonomous third order differential equations of the forms

$$\frac{\dddot{x}}{\dot{x}^3} - \frac{3}{2}\frac{\ddot{x}^2}{\dot{x}^4} = -\frac{1}{2}\frac{x^2 - x + 1}{x^2(x-1)^2}, \qquad \frac{\dddot{y}}{\dot{y}^3} - \frac{3}{2}\frac{\ddot{y}^2}{\dot{y}^4} = \boldsymbol{\mathcal{Q}}(x,y)$$

with the explicitly calculable rational function $\boldsymbol{\mathcal{Q}}(x,y)$ for all (n,m,N).

Because of this proposition and the simplicity of formulae (33), there is no need to use the equations $P(x,y) = 0$ because the uniformization allows investigating solutions with arbitrarily large (n,m,N). Here, we include differentiation, the Laurent–Puiseux series, calculation of monodromies, the branching, the genus, plotting of graphs, etc.

The zero genus is not a general property of the solutions under study. For example, under $A\tau + B = \frac{1}{5}$ the Picard solution has the form of a cumbersome but elliptic curve $P(\overset{6}{x}, \overset{12}{y}) = 0$:

$$\begin{aligned} &y^{12} - 50xy^{10} + 140(x^2+x)y^9 - 5(32x^3 + 89x^2 + 32x)y^8 \\ &\quad + 16(4x^4 + 35x^3 + 35x^2 + 4x)y^7 - 60(4x^4 + 13x^3 + 4x^2)y^6 + 360(x^4 + x^3)y^5 \\ &\quad - 105x^4y^4 - 80(x^5 + x^4)y^3 + 2(8x^6 + 47x^5 + 8x^4)y^2 - 20(x^6 + x^5)y + 5x^6 = 0 \end{aligned} \tag{34}$$

with the Klein modular invariant $J = \frac{256}{135}$. The Weierstrass $\wp$-parameterization for (34) is very complicated and less adjustable for this curve, while for any $\tau \in \mathbb{H}^+$ we have

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \qquad y = -\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)} \frac{\theta_2^2\big(\frac{1}{10}\big|\tau\big)}{\theta_1^2\big(\frac{1}{10}\big|\tau\big)}.$$

[4]In particular, $\theta_1'(p\tau + q|\tau)$ is algebraically related to the functions $\theta(p\tau + q|\tau)$, for rational p, q while this relation can be only differential in the generic case (see Lemma 1 in the appendix).

The curve genus increases rapidly. For $N = 4, 5, 6, 7, 8, 9, \ldots$ the genera of the respective Picard–Hitchin solutions are $g = 0, 1, 1, 4, 5, 7, \ldots$. We mention here that the genus depends on the choice of the representing equation with the Painlevé property. For instance, the same solution (34), but in Picard representation (23–24) has a nonhyperelliptic genus $g = 3$, and a simple solution of equation (24), which is birationally isomorphic to the lemniscate $w^2 = z^4 - 1$ (the calculation is nontrivial), already appears at $A\tau + B = \frac{1}{4}$:

$$\begin{aligned} x^4u^{16} - 20x^3u^{12} + 32x^2(x+1)u^{10} - 2x(8x^2+29x+8)u^8 \\ + 32x(x+1)u^6 - 20xu^4 + 1 = 0. \end{aligned} \tag{35}$$

Remark 2. In [4], we advanced an assumption about a possible relation between the theta constants $\theta\big(u(\tau)|\tau\big)$ of general form and the uniformizing Fuchs equations. A particular case of such a constant is just the Dedekind function

$$\boldsymbol{\eta}(\tau) = -\mathrm{i}\mathrm{e}^{\frac{\pi\mathrm{i}}{3}\tau}\theta_1(\tau|3\tau),$$

which appears both in our formulae and in some known but isolated parameterizations of modular equations (mostly of the genus zero; see the bibliography in [4]). It follows from the above results that the Painlevé equation admits an infinite class of generalizations in this direction using the fundamental object $\theta\big(\frac{n}{N}\tau + \frac{m}{N}|\tau\big)$, together with the explicit formulae for curves that are not necessarily solutions of the equation $\mathcal{P}_6$, for functions on these curves, for the differential equations on these functions corresponding to the subgroups of $\boldsymbol{\Gamma}(2)$, and so on.

We note the following computational fact. The Painlevé curves are specific and not always easily accessible by various algorithms. Curve (35) is a good example. But in the framework of what was said above, convenient equations appear, which in particular include an important class of hyperelliptic curves. Even less obvious is that function (31) in fact corresponds to a larger curve

$$\boldsymbol{z}^2 = (\boldsymbol{x}^8 + 14\boldsymbol{x}^4 + 1)(\boldsymbol{x}^5 - \boldsymbol{x}). \tag{36}$$

Namely, function (31) also parameterizes this equation in the sense that $\boldsymbol{z}(\tau)$, determined from Eqs. (31) and (36) is an analytic function that is uniquely defined in its whole domain, i.e., in $\mathbb{H}^+$. This example provides one more realization of a rather nontrivial tower of embedded hyperelliptic curves with the universal uniformizing function $\chi(\tau)$ (see [4] about this function).

We omit both the derivation of formula (36) and other theta-functional consequences of what was said above. Because of their large number, they deserve a separate investigation that is beyond the scope of this paper. This material is presented in work [6].

6. Concluding remarks

Using the analytic results in both x-representation (15), (17), and τ-representation (12), (18) we can easily obtain several useful formulae, for example, simple expressions for the conformal factor F of the Tod–Hitchin metric [3, 13, 25]. This factor was expressed in terms of the derivatives of the theta functions with respect to the integration constants in [3, theorem 5.1], and the solution $y(x)$ was expressed in terms of the double derivatives

of the theta functions with respect to both its arguments. By Lemma 1 the final answers are actually expressed in the basis $\vartheta, \theta_1', \theta_k(A\tau + B|\tau)$. Another example is third-order differential equations for the $\boldsymbol{\tau}$-functions and for the entire holomorphic functions whose ratio gives the solution $y(x)$. On numerous occasions, Painlevé himself indicated the existence and importance of these equations (see, e.g., [20, p. 1114]). They follow from the third-order equations for $\sigma(a\tau + b|\tau)$, which follow from Lemma 3 in the appendix.

Recalling the remark in sect 1.1, we note that the known solutions and local branchings [11, 16, 13, 12] are compatible with the uniformization by the Painlevé substitution, i.e., they generate the τ-representations of the solutions in terms of the single-valued functions $y(\tau)$. The necessary data for establishing this property for all the solutions is presumably contained in Theorem 2 in [18] and in results of work by Guzzetti [12]. But in the uniformization context, it is desirable to have the direct statement about the global single-valuedness of the function $\wp\big(z(\tau)|\tau\big)$ for all values of $(\alpha, \beta, \gamma, \delta)$. This would provide a new important interpretation of the equation $\mathcal{P}_6$. Equation (1) would differ from the lower Painlevé equations in that their solutions are globally meromorphic on the plane $\mathbb{C}$; for the equation $\mathcal{P}_6$, they are globally meromorphic on the upper half-plane $\mathbb{H}^+$ with the constant negative Lobachevskii curvature, which is the universal covering of $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$. A broad class of algebraic solutions is then provided by automorphic functions on the Riemann surfaces with three punctures (orbifolds), while the transcendental solutions of the Picard–Hitchin type themselves generate another, physically meaningful class of single-valued functions. In this context, solutions to equation (2) do not follow automatically from those of (1), and integrating (2) is therefore an independent problem.

7. Appendix: rules for differentiating the theta-function

7.1. Jacobi's θ-functions. These functions are used in the following definitions [1]:

$$\theta_1(z|\tau) = -\mathrm{i}\mathrm{e}^{\frac14\pi\mathrm{i}\tau} \sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau} \mathrm{e}^{(2k+1)\pi\mathrm{i}z}, \qquad \theta_3(z|\tau) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{k^2\pi\mathrm{i}\tau} \mathrm{e}^{2k\pi\mathrm{i}z},$$

$$\theta_2(z|\tau) = \mathrm{e}^{\frac14\pi\mathrm{i}\tau} \sum_{k=-\infty}^{\infty} \mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau} \mathrm{e}^{(2k+1)\pi\mathrm{i}z}, \qquad \theta_4(z|\tau) = \sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{k^2\pi\mathrm{i}\tau} \mathrm{e}^{2k\pi\mathrm{i}z}.$$

We also introduce the fifth independent object

$$\theta_1'(z|\tau) = \pi\mathrm{e}^{\frac14\pi\mathrm{i}\tau} \sum_{k=-\infty}^{\infty} (-1)^k (2k+1) \mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau} \mathrm{e}^{(2k+1)\pi\mathrm{i}z}$$

and the ϑ-constants are defined as $\vartheta_k := \vartheta_k(\tau) = \theta_k(0|\tau)$ and $\vartheta_1 \equiv 0$.

Lemma 1. *The five Jacobi functions $\theta_1, \theta_2, \theta_3, \theta_4$ and $\theta_1' := \dfrac{\partial\theta_1}{\partial z}$ satisfy the closed ordinary differential equations with respect to the variables (z, τ) over the field of coefficients*

$\eta(\tau)$ and $\vartheta^2(\tau)$:

$$\left\{\begin{aligned} \frac{\partial\theta_k}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_k - \pi\vartheta_k^2\cdot\frac{\theta_\nu\theta_\mu}{\theta_1}\\ \frac{\partial\theta_1'}{\partial z} &= \frac{\theta_1'^{\,2}}{\theta_1} - \pi^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} - 4\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_1 \end{aligned}\right. , \tag{37}$$

$$\left\{\begin{aligned} \frac{\partial\theta_k}{\partial \tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^{\,2}}{\theta_1^2}\theta_k + \frac{\mathrm{i}}{2}\vartheta_k^2\cdot\theta_1'\frac{\theta_\nu\theta_\mu}{\theta_1^2} + \frac{\pi\mathrm{i}}{4}\Big\{\vartheta_3^2\vartheta_4^2\cdot\theta_2^2 - \vartheta_k^2\vartheta_\mu^2\cdot\theta_\nu^2 - \vartheta_k^2\vartheta_\nu^2\cdot\theta_\mu^2\Big\}\frac{\theta_k}{\theta_1^2}\\ &\qquad + \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_k\\ \frac{\partial\theta_1'}{\partial \tau} &= \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^{\,3}}{\theta_1^2} + \frac{3\mathrm{i}}{\pi}\Big\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1^2} + \eta + \frac{\pi^2}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\theta_1' - \frac{\pi^2}{2}\mathrm{i}\vartheta_2^2\vartheta_3^2\vartheta_4^2\cdot\frac{\theta_2\theta_3\theta_4}{\theta_1^2} \end{aligned}\right. ,$$

where

$$\nu = \frac{8k-28}{3k-10}, \qquad \mu = \frac{10k-28}{3k-8}, \qquad k = 1,2,3,4.$$

These differentiations are not completely symmetric, and the equations themselves mean that not only elliptic functions (the ratios θ_j/θ_k) have the closed differential calculus with respect to both the variables (z,τ) independently but also the same holds for their 'constituents', the theta functions including the function θ_1'. The details and, not less important, the analysis of the integrability condition for the above equations have been detailed in work [5]. The basis $\theta_{1,2,3,4}$ can be obviously closed by adding any of the functions θ_k', not necessarily θ_1'. It is essential that the known equation $4\pi\mathrm{i}\theta_\tau = \theta_{zz}$ must be treated as a corollary of the above equations and not vice versa because the Lemma 1 defines ordinary differential equations, while this equation is an equation of the heat kernel type in partial derivatives; the theta functions are just particular solutions of it.

Lemma 2. *Let n be an integer and $n_1 := n-1$. Then the Jacobi functions satisfy the recursive multiplication formulae*

$$\left\{\begin{aligned} \theta_1(2z) &= 2\theta_1(z)\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\vartheta_2\vartheta_3\vartheta_4}\\ \theta_1(nz) &= \frac{\theta_3^2(n_1z)\theta_2^2(z) - \theta_2^2(n_1z)\theta_3^2(z)}{\vartheta_4^2\cdot\theta_1\big((n-2)z\big)} \end{aligned}\right. , \tag{38}$$

$$\left\{\begin{aligned} \theta_2(nz) &= \frac{\theta_3^2(n_1z)\theta_3^2(z) - \theta_4^2(n_1z)\theta_4^2(z)}{\vartheta_2^2\cdot\theta_2\big((n-2)z\big)}\\ \theta_3(nz) &= \frac{\theta_2^2(n_1z)\theta_2^2(z) + \theta_4^2(n_1z)\theta_4^2(z)}{\vartheta_3^2\cdot\theta_3\big((n-2)z\big)}\\ \theta_4(nz) &= \frac{\theta_3^2(n_1z)\theta_3^2(z) - \theta_2^2(n_1z)\theta_2^2(z)}{\vartheta_4^2\cdot\theta_4\big((n-2)z\big)} \end{aligned}\right. .$$

We obtain the multiplication formula for the function $\theta_1'(nz)$ by taking the derivative in (38), and subsequently using formulae (37). The lemma statement also holds for an arbitrary complex n.

It is natural here to assume that there are general nonrecursive expressions for these formulae in form of certain determinants.

7.2. **The Weierstrass functions.** The modular Weierstrass functions $g_2(\tau)$, $g_3(\tau)$ and $\eta(\tau)$ are defined by the series

$$\begin{aligned}
\eta(\tau) &= 2\pi^2\left\{\frac{1}{24}-\sum_{k=1}^{\infty}\frac{\mathrm{e}^{2k\pi\mathrm{i}\tau}}{\big(1-\mathrm{e}^{2k\pi\mathrm{i}\tau}\big)^2}\right\},\\
g_2(\tau) &= 20\pi^4\left\{\frac{1}{240}+\sum_{k=1}^{\infty}\frac{k^3\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}=\frac{\pi^4}{24}\big(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8\big),\\
g_3(\tau) &= \frac{7}{3}\pi^6\left\{\frac{1}{504}-\sum_{k=1}^{\infty}\frac{k^5\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}=\frac{\pi^6}{432}\big(\vartheta_2^4+\vartheta_3^4\big)\big(\vartheta_3^4+\vartheta_4^4\big)\big(\vartheta_4^4-\vartheta_2^4\big).
\end{aligned}\tag{39}$$

Lemma 3. *The Weierstrass functions $\sigma,\zeta,\wp,\wp'(z|\tau)$ satisfy the nonautonomous dynamical system with the coefficients $\eta(\tau)$, $g_2(\tau)$ and with the parameter z:*

$$\left\{\begin{aligned}
\frac{\partial\sigma}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp-\zeta^2+2\eta(z\zeta-1)-\frac{1}{12}g_2z^2\Big\}\sigma\\
\frac{\partial\zeta}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp'+2(\zeta-z\eta)\wp+2\eta\zeta-\frac{1}{6}g_2z\Big\}\\
\frac{\partial\wp}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\Big\{2(\zeta-z\eta)\wp'+4(\wp-\eta)\wp-\frac{2}{3}g_2\Big\}\\
\frac{\partial\wp'}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\Big\{6(\wp-\eta)\wp'+(\zeta-z\eta)(12\wp^2-g_2)\Big\}
\end{aligned}\right\}\ ,\tag{40}$$

where we use the right brace additionally to denote the differential closedness of the functions $\zeta,\wp,\wp'$. Using the replacements $\zeta\mapsto\zeta-z\eta$ and $\sigma\mapsto\sigma\exp\{-\frac{1}{2}\eta z^2\}$ we eliminate the parameter z from Eqs. (40), which is equivalent to passing to the functions θ' and θ.

Because $\wp'$ can be expressed algebraically in terms of $\wp$, the 'essential' part of system (40) is its second and third equations. In turn, they are equivalent to the equations for the functions $\zeta(A\tau+B|\tau)$ and $\wp(A\tau+B|\tau)$, if we take the known relation $\partial_z\zeta(z|\tau)=-\wp(z|\tau)$ into account. Therefore, if we regard the parameters $(\alpha,\beta,\gamma,\delta)$ as the moduli of the equation $\mathcal{P}_6$, then the Picard–Hitchin–Okamoto class (see Remark 1) has common moduli, and all the equations $\mathcal{P}_6$ in this class are then equivalent to a single representative, which is system (40).

Corollary 3. *The canonical representative of the equations $\mathcal{P}_6$ in the class of the Picard–Hitchin–Okamoto parameters is the system of two equations for the functions $\boldsymbol{\zeta}$, $\boldsymbol{\wp}$:*

$$\frac{d\boldsymbol{\zeta}}{d\tau}=\frac{\mathrm{i}}{\pi}\Big\{\boldsymbol{\wp}'+2(\boldsymbol{\wp}+\eta)\boldsymbol{\zeta}\Big\},\qquad \frac{d\boldsymbol{\wp}}{d\tau}=-\frac{\mathrm{i}}{\pi}\Big\{2\boldsymbol{\zeta}\boldsymbol{\wp}'+4(\boldsymbol{\wp}-\eta)\boldsymbol{\wp}-\tfrac{2}{3}g_2\Big\}\tag{41}$$

and its common integral

$$\boldsymbol{\zeta}=\zeta(A\tau+B|\tau)-A\eta'(\tau)-B\eta(\tau),\qquad \boldsymbol{\wp}=\wp(A\tau+B|\tau).$$

Here $\boldsymbol{\wp}'=\sqrt{4\boldsymbol{\wp}^3-g_2\boldsymbol{\wp}-g_3}$, *and Eqs.* (41) *are supplemented by their corollary*:

$$\frac{d\boldsymbol{\wp}'}{d\tau}=-\frac{\mathrm{i}}{\pi}\Big\{6(\boldsymbol{\wp}-\eta)\boldsymbol{\wp}'+(12\boldsymbol{\wp}^2-g_2)\boldsymbol{\zeta}\Big\}.$$

It is clear that we can keep either of the quantities $\boldsymbol{\wp}$, $\boldsymbol{\wp}'$, and solutions and their derivatives are then rational functions of $(\boldsymbol{\zeta},\boldsymbol{\wp},\boldsymbol{\wp}')$. Such derivatives can then be treated as the $\wp$-form of Okamoto transformations (20). For instance, for the Hitchin solution $H=\wp(z|\tau)$, we have

$$H=\boldsymbol{\wp}+\frac{1}{2}\frac{\boldsymbol{\wp}'}{\boldsymbol{\zeta}},\qquad \frac{\pi}{\mathrm{i}}\frac{dH}{d\tau}=\frac{4H^3-g_2H-g_3}{\boldsymbol{\wp}-H}+2H^2+4\eta H+\frac{1}{6}g_2.$$

Choosing a convenient basis composed of $(\boldsymbol{\zeta},\boldsymbol{\wp},\boldsymbol{\wp}')$ based on governing equations (41), we can go further and develop the inverse, i.e., the integral Painlevé calculus in both the uniformizing (τ)-half-plane and the (x)-plane. There are numerous examples, and in addition to the first obvious example following from the $\boldsymbol{\tau}$-form of (15)

$$\int^x\frac{y\,dx}{x(x-1)}=-2\,\mathrm{Ln}\frac{\theta_1'\big(A\frac{K}{K'}+B\big|\frac{\mathrm{i}K}{K'}\big)+2\pi A\cdot\theta_1\big(A\frac{K}{K'}+B\big|\frac{\mathrm{i}K}{K'}\big)}{\sqrt{1-x}\,K'\cdot\theta_1\big(A\frac{K}{K'}+B\big|\frac{\mathrm{i}K}{K'}\big)},$$

we present another, more remarkable example:

$$\frac{\mathrm{i}}{\pi}\int^\tau(\boldsymbol{\wp}-\boldsymbol{\zeta}^2)\,d\tau=\mathrm{Ln}\,\theta_1\big(\tfrac12A\tau+\tfrac12B|\tau\big)-\mathrm{Ln}\,\boldsymbol{\eta}(\tau)+\frac{\pi\mathrm{i}}{4}A^2\tau.$$

This example, which generates all other integrals, can be logically considered the Painlevé (τ)-analogue of the well-known integral Weierstrass relation between the meromorphic objects ζ, $\wp$ and the entire function σ. Again imposing condition (27), we obtain various theta constants $\theta\big(u(\tau)|\tau\big)$, $\theta'\big(u(\tau)|\tau\big)$ of the general form in the left-hand side and the value of the integral calculated on these constants, itself a theta constant, in the right-hand side.

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